General Matrix-Valued Inhomogeneous Linear Stochastic Differential Equations and Applications

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Abstract: The expressions of solutions for general $n \times m$ matrix-valued inhomogeneous linear stochastic differential equations are derived. This generalizes a result of Jaschke (2003) for scalar inhomogeneous linear stochastic differential equations. As an application, some $I\!R^n$ vector-valued inhomogeneous nonlinear stochastic differential equations are reduced to random differential equations, facilitating pathwise study of the solutions.

1 A Review of Stochastic Exponential Formulas

We first review some existing results about solution formulas for linear stochastic differential equations (SDEs) or for their integral formulations. Let $(\Omega, \mathcal{F}, (\mathcal{F}t), \mathbf{P})$ be a standard stochastic basis. For the following stochastic integral equation

$$X_t = 1 + \int_0^t X_{s-} dZ_s, \qquad (1.1),$$

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where Z is a semimartingale with $Z_0 = 0$, Doléans-Dade (1970) proved that the unique solution of (1.1) is given by

$$X_t = \exp\Big\{Z_t - \frac{1}{2}\langle Z^c, Z^c\rangle_t\Big\} \prod_{0<s\leq t} (1 + \Delta Z_s)e^{-\Delta Z_s}. \tag{1.2}$$

In the literature X is called *Doléans exponential (or stochastic exponential)* of Z, and is denoted by $\mathcal{E}(Z)$. The formula (1.2) is called the *Doléans (or stochastic) exponential formula.*

In an unpublished paper, Yoeurp and Yor (1977) proved the following result for the solution formula of scalar SDEs (see also Revuz-Yor (1999) and Protter (2005) for the case where Z is a continuous semimartingale, and Melnikov-Shiryaev (1996)) for the general case).

Theorem 1.1 ***(Yoeurp and Yor 1977*** *) Let Z and H be semimartingales, and $\Delta Z_s \neq -1$ for all $s \in [0, \infty]$. Then the unique solution of the inhomogeneous scalar linear SDE*

$$X_t = H_t + \int_0^t X_{s-} dZ_s, \quad t \geq 0, \tag{1.3}$$

is given by

$$X_t = \mathcal{E}(Z)_t\Big\{H_0 + \int_0^t \mathcal{E}(Z)_{s-}^{-1} dG_s\Big\}, \tag{1.4}$$

where

$$G_t = H_t - \langle H^c, Z^c\rangle_t - \sum_{0<s\leq t} \frac{\Delta H_s \Delta Z_s}{1 + \Delta Z_s}. \tag{1.5}$$

Jaschke (2003) extended equation (1.3) to the case where H is an adapted cadlag process, not necessarily a semimartingale. He proved that in this case the solution of (1.3) is given by:

$$X_t = H_t - \mathcal{E}(Z)_t \int_0^t H_{s-} d(\mathcal{E}(Z)_s^{-1}). \tag{1.6}$$

By using (1.6) Jaschke (2003) obtained a new proof of (1.4).

On the other hand, Emery (1978) considered the following $n\times n$ matrix-valued stochastic equation

$$U(t) = I + \int_0^t (dL(s))U(s-), \qquad (1.7),$$

where I is an $n\times n$ identity, L is a given $n\times n$ matrix-valued semimartingale with $L(0) = 0$, such that $I + \Delta L(s)$ is invertible for all $s \in [0, \infty]$, where $\Delta L(s) = L(s) - L(s-)$. Emery proved that the equation (1.7) admits a unique solution U, which is $n \times n$ matrix-valued semimartingale. We call it the stochastic exponential of L and denote it by $\mathcal{E}(L)$. However, there is no explicit expression for such a stochastic exponential in general.

Jacod (1982) has studied the following inhomogeneous matrix-valued stochastic integral equation

$$X(t) = H(t) + \int_0^t (dL(s))X(s-), \tag{1.8),}$$

where L is a given $n \times n$ matrix-valued semimartingale with $L0 = 0$, such that $I + \Delta L(s)$ is invertible for all $s \in [0, \infty]$, and H is an $n \times m$ matrix valued semimartingale. For an $n \times n$ matrix valued semimartingale A and an $n \times m$ matrix valued semimartingale B, we let

$$[A, B](t) = \langle A^c, B^c \rangle(t) + \sum_{0<s\leq t} \Delta A(s) \Delta B(s).$$

Here A^c denotes its continuous martingale part defined componentwise by $(L^c)^i_j = (L^i_j)^c$, and

$$\langle A^c, B^c \rangle^i_j = \sum_k \langle (A^c)^i k, (B^c)^k_j \rangle. \tag{1.9}$$

Using these notations the result of Jacod (1982) implies the following

Theorem 1.2 ***(Jacod 1982)*** *The unique solution of (1.8) is given by*

$$X(t) = \mathcal{E}(L)(t)\Big\{H(0) + \int_0^t \mathcal{E}(L)(s-)^{-1} dG(s)\Big\}, \tag{1.10}$$

where

$$G(t) = H(t) - \langle L^c, H^c \rangle(t) - \sum_{0<s\leq t} (1 + \Delta L(s))^{-1} \Delta L(s) \Delta H(s). \tag{1.11}$$

In particular, if L and H are continuous semimartingales, an expression for the solution of (1.8) is given in Revuz-Yor (1999) as follows:

$$X(t) = \mathcal{E}(L)(t)\Big\{H(0) + \int_0^t \mathcal{E}(L)(s)^{-1}(dH(s) - d[L, H](s))\Big\}. \tag{1.12}$$

The objective of the present note is to generalize equation (1.8) to the case where $H(t)$ is a given $n \times m$ matrix-valued adapted cadlag process, not necessarily a semimartingale. We give an expression of the solution of (1.8) for this case and also give a simpler proof for Theorem 1.1. Our result extends (1.6) of Jaschke (2003) to matrix-valued case. As an application, we reduce some $I\!R^n$-valued inhomogeneous *nonlinear* SDEs to random differential equations (RDEs) — differential equations with random coefficients. This facilitates pathwise study of solutions and is an important step in the context of random dynamical systems approaches; see Arnold (1998).

2 Matrix-valued Inhomogeneous Linear SDEs

Léandre (1985) obtained the following result about stochastic equation (1.7). If we denote by V the inverse of $\mathcal{E}(L)$, then V is the solution of the following equation:

$$V(t) = I + \int_0^t V(s-)dW(s),$$

where

$$W(t) = -L(t) + \langle L^c, L^c\rangle(t) + \sum_{0<s\le t}(1+\Delta L(s))^{-1}(\Delta L(s))^2.$$

That means $\mathcal{E}(L)^{-1} = \mathcal{E}(W^\tau)^\tau$. We refer the reader to Karandikar (1991) for a detailed proof of this result.

Now we will use this result of Léandre (1985) to solve the following inhomogeneous stochastic integral equation

$$X(t) = H(t) + \int_0^t (dL(s))X(s-), \qquad (2.1),$$

where L is a given $n \times n$ matrix-valued semimartingale with $L0 = 0$, such that $I + \Delta L(s)$ is invertible for all $s \in [0, \infty]$, $H(t)$ is a given $n \times m$ matrix-valued adapted cadlag process.

Our main result is the following.

Theorem 2.1 *The unique solution of (2.1) is given by*

$$X(t) = H(t) - \mathcal{E}(L)(t)\int_0^t (d\mathcal{E}(L)(s)^{-1})H(s-). \qquad (2.2)$$

If H is $n \times m$ matrix-valued semimartingale, then $X(t)$ has the same expression as given by (1.10) and (1.11).

Proof. We denote $\mathcal{E}(L)$ and $\mathcal{E}(L)^{-1}$ by U and V, respectively. We are going to show that the process

$$X(t) = H(t) - U(t)\int_0^t (dV(s))H(s-),$$

defined by (2.2), satisfies equation (2.1). Since $UV = I$, by the integration by parts formula (see Karandikar (1991)) we get

$$\begin{aligned} 0 &= d(U(t)V(t)) = U(t-)dV(t) + (dU(t))V(t-) + d[U,V](t) \\ &= d(U(t)V(t)) = U(t-)dV(t) + dL(t) + d[U,V](t). \end{aligned}$$

Once again by the integration by parts formula, using the above result and the fact that $dU(t) = (dL(t))U(t-)$, we have

$$\begin{aligned} d(X(t) - H(t)) &= -U(t-)dV(t)H(t-) - (dU(t))(\int_0^{t-} dV(s)H(s-)) - (d[U,V](t))H(t-) \\ &= (dL(t))[H(t-) - U(t-)(\int_0^{t-} dV(s)H(s-))] = (dL(t))X(t-). \end{aligned}$$

This shows that the process $(X(t))$ defined by (2.2) satisfies (2.1).

Now we assume that $(H(t))$ is an $n \times m$ matrix-valued semimartingale. Using the notations in Section 1 we can verify that

$$G(t) = H(t) + [W, H](t). \tag{2.3}$$

By the integration by parts formula, using (2.3) and the fact that $dV(t) = V(t-)dW(t)$, we obtain

$$\begin{aligned} 0 &= d(V(t)H(t)) = V(t-)dH(t) + (dV(t))H(t-) + V(t-)d[W, H](t) \\ &= V(t-)dG(t) + dL(t) + (dV(t))H(t-), \end{aligned}$$

from which we get

$$H(t) = U(t)\Big\{H(0) + \int_0^t V(s-)dG(s) + \int_0^t dV(s)H(s-)\Big\}.$$

Thus, if we let $(G(t))$ be defined by (2.3) then $(X(t))$ has the expression of (2.2), and consequently it satisfies equation (2.1). The proof of the theorem is complete. □

3 An Application to Nonlinear SDEs

In this section we will apply our results in Theorem 2.1 to reduce an inhomogeneous *nonlinear* SDE to a RDE (random differential equation). Now we consider the following n-dimensional nonlinear SDE (but with a linear multiplicative noise term):

$$dX^i(t) = f^i(t, X(t))dt + \sum_{j=1}^n C^i_j(t)X^j(t)dB^j(t), \quad X^i(0) = x^i, \tag{3.1}$$

where $C(t)$ is an $n \times n$ matrix-valued measurable function, $f(t, x)$ is a $I\!R^n$-valued measurable function on $[0, \infty) \times I\!R^n$, and $B(t) = (B^1(t), \cdots, B^n(t))^\tau$ is a n-dimensional Brownian motion. Put

$$L^i_j(t) = \int_0^t C^i_j(s)dB^j(s), \ i.j = 1, \cdots, n; \quad H(t) = X(0) + \int_0^t f(s, X(s))ds. \tag{3.2}$$

Then (3.1) can be rewritten in the form of (2.1). For such L, the equation (1.9) is reduced to the following linear equation:

$$dU^i_j(t) = \sum_{k=1}^n C^i_k(t)U^k_j(t)dB^k(t), \quad U^i_j(0) = \delta^i_j, \tag{3.3}$$

In the present case we have $G(t) = H(t)$. So according to (2.2), the solution of (3.1) can be expressed as

$$X(t) = U(t)\Big\{x + \int_0^t U(t)^{-1}f(s, X(s))ds\Big\}, \tag{3.4}$$

where U is the solution of (3.3). Unfortunately, even in this case we are not able to give an explicit expression for $U(t)$. Let $Y(t) = U(t)^{-1}X(t)$, then

$$Y(t) = \Big\{x + \int_0^t U(s)^{-1} f(s, U(s)Y(s))ds\Big\}. \tag{3.5}$$

This is the integral formulation of a RDE (random differential equation). Once we have sample path solution $Y(t)$ of this transformed RDE , we obtain the solution of the original SDE via $X(t) = U(t)Y(t)$.

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